Boundary conditions in electromagnetics

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A proof is given to show that satisfying the boundary conditions on the tangential electric and magnetic fields across the boundary surface of two different material media means that the boundary conditions on the normal magnetic-induction vector and electric-displacement vector are automatically satisfied while the converse is not true.

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Boundary conditions for electromagnetic fields are the cornerstones for classical theory of electrodynamics. Their derivations are standard textbook material [1–5]. For example, the boundary conditions for electromagnetic fields at the boundary of two distinct dielectric media, derived through the application of Stoke's theorem to Maxwell's curl equations over a rectangular area which borders the two boundary media, are

$$\mathbf{n}\times\mathbf{E}_1=\mathbf{n}\times\mathbf{E}_2 , \tag{1}$$

$$\mathbf{n}\times\mathbf{H}_1=\mathbf{n}\times\mathbf{H}_2 , \tag{2}$$

where $\mathbf{n}$ is a unit vector normal to the boundary, $\mathbf{E}_1$ and $\mathbf{E}_2$ are, respectively, the electric field in medium 1 and in medium 2 at the boundary, and $\mathbf{H}_1$ and $\mathbf{H}_2$ are, respectively, the magnetic field in medium 1 and in medium 2 at the boundary; the boundary conditions derived through the application of divergence theorem to Maxwell's divergence equations over a pill box which borders the two boundary media, are

$$\mathbf{n}\cdot\mathbf{D}_1=\mathbf{n}\cdot\mathbf{D}_2 , \tag{3}$$

$$\mathbf{n}\cdot\mathbf{B}_1=\mathbf{n}\cdot\mathbf{B}_2 , \tag{4}$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are, respectively, the displacement vector in medium 1 and in medium 2 at the boundary, while $\mathbf{B}_1$ and $\mathbf{B}_2$ are, respectively, the magnetic flux density in medium 1 and in medium 2 at the boundary.

It is important to note that the boundary conditions on the normal components of $\mathbf{D}$ and $\mathbf{B}$ [Eqs. (3) and (4)] are redundant, since, according to the uniqueness theorem [1], the boundary conditions on the tangential components of $\mathbf{E}$ and $\mathbf{H}$ [Eqs. (1) and (2)] are necessary and sufficient boundary conditions. In other words, satisfying Eqs. (1) and (2) implies that Eqs. (3) and (4) are automatically satisfied while the converse is not true. Although this fact is well known, no proof has been found in the standard literature. It is the purpose of this paper to present this proof.

Across the boundary, as shown in Fig. 1, let us introduce two small parallel surface areas of rectangular shape that are mirror images of each other. The top rectangle, parallel to the interface, is located in medium 1, characterized by $(\epsilon_1,\mu_1,\sigma_1)$ where ϵ_1, μ_1, and σ_1 are, respectively, the permittivity, the permeability, and the conductivity of medium 1, while the bottom rectangle, also parallel to the interface, is located in medium 2, characterized by $(\epsilon_2,\mu_2,\sigma_2)$ where ϵ_2,μ_2, and σ_2 are, respectively, the permittivity, the permeability, and the conductivity of medium 2. The small rectangle has sides Δs_1 and Δs_2. The unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to the rectangular surfaces as shown in Fig. 1. The vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the three unit vectors in the x,y,z directions, respectively.

The source-free Maxwells equations are

$$\nabla\times\mathbf{E}=-\frac{\partial\beta}{\partial t} , \tag{5}$$

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t} , \tag{6}$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{D}$ refer to the field quantities in the medium in which they apply. Integrating Eq. (5) over the rectangular area ΔS_1 in region 1 yields

$$\int_{\Delta S_1}(\nabla\times\mathbf{E}_1)\cdot\mathbf{n}_1 dS=-\frac{\partial}{\partial t}\int_{\Delta S_1}\mathbf{B}_1\cdot\mathbf{n}_1 dS , \tag{7}$$

and integrating Eq. (5) over the rectangular area ΔS_2 in region 2 yields

$$\int_{\Delta S_2}(\nabla\times\mathbf{E}_2)\cdot\mathbf{n}_2 dS=-\frac{\partial}{\partial t}\int_{\Delta S_2}\mathbf{B}_2\cdot\mathbf{n}_2 dS . \tag{8}$$

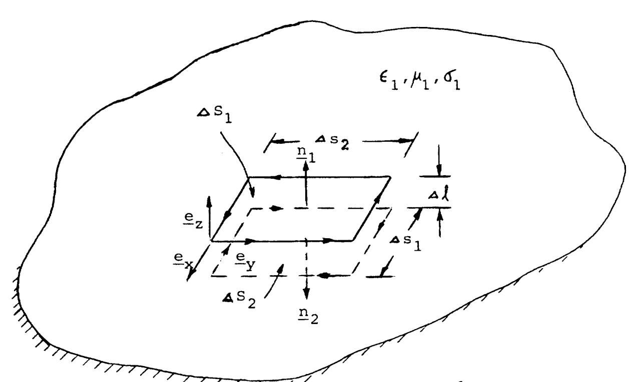

FIG. 1. Geometry of the problem. Rectangular area ΔS_1 is parallel to rectangular area ΔS_2. Sides of rectangle are Δs_1 and Δs_2. Δl is the separation between the two rectangles.

Application of Stoke's theorem to Eqs. (7) and (8) and adding the resultant equations yield

$$\int_{c_1} \mathbf{E}_1\cdot d\mathbf{s}_1 + \int_{c_2} \mathbf{E}_2\cdot d\mathbf{s}_2 = -\frac{\partial}{\partial t}\left[\int_{\Delta S_1} \mathbf{B}_1\cdot\mathbf{n}_1 dS + \int_{\Delta S_2} \mathbf{B}_2\cdot\mathbf{n}_2 dS\right], \quad (9)$$

where c_1 and c_2 are, respectively, the circumferences of the rectangular areas ΔS_1 and ΔS_2. In rectangular coordinates, with $\Delta s_1 \to 0$ and $\Delta s_2 \to 0$, **E** has a constant value along each side. Allowing the separation Δl between the two parallel rectangular areas in medium 1 and in medium 2 to approach zero, one has

$$(E_{1x}-E_{2x})2\Delta x + (E_{1y}-E_{2y})2\Delta y = -\frac{\partial}{\partial t}(B_{1z}-B_{2z})\Delta x\,\Delta y\ . \quad (10)$$

From Fig. 1, one may immediately identify the following:

$$\Delta s_1 = \Delta x,\quad \Delta s_2 = \Delta y\ ,$$

$$\mathbf{s}_1 = \mathbf{e}_x,\quad \mathbf{s}_2 = \mathbf{e}_y\ ,$$

$$\mathbf{n}_1 = -\mathbf{n}_2 = \mathbf{n} = \mathbf{e}_z\ .$$

The quantities E_{1x} and E_{2x} are, respectively, the x-directed electric field tangential to the boundary in region 1 and in region 2, the quantities E_{1y} and E_{2y} are, respectively, the y-directed electric field tangential to the boundary in region 1 and in region 2, and the quantities B_{1z} and B_{2z} are, respectively, the z-directed **B** field normal to the boundary surface in region 1 and in region 2.

Equation (10) shows that if the condition that the tangential components of the electric field are continuous across the boundary is satisfied, i.e., $E_{1x}=E_{2x}$ *and* $E_{1y}=E_{2y}$, then it follows that $B_{1z}=B_{2z}$, i.e., the condition that the normal component of the **B** field is continuous across the boundary is automatically satisfied. But, if the condition that the normal component of **B** field is continuous across the boundary is satisfied, i.e., $B_{1z}=B_{2z}$, according to Eq. (10) the two boundary conditions ($E_{1x}=E_{2x}$, $E_{1y}=E_{2y}$) on the two components of the tangential electric field are *not necessarily* satisfied.

A similar proof can be made for the tangential components of **H** and the normal component of **D**, using Eq. (6).

It is therefore proven that satisfying the boundary conditions on the tangential electric and magnetic fields across the boundary surface implies that the boundary conditions on the normal magnetic induction vector and the normal electric displacement vector are satisfied while the converse is not true.

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